A-T-menability of groups

A Thesis in

Mathematics

by

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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Master's degree Plan B

April 2011

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Abstract

This paper presents a detailed study of a-T-menable discrete groups. Starting with several conditions required for a-T-menability, we prove that they are equivalent and hence charaterize a class of a-T-menable discrete groups. We then show that the free groups on two generators is a-T-menable. Using the infinite cyclic group, we succesfully draw a rigid connection – from the perspective of affine isometric actions – between amenable groups and a-T-menable groups. We also prove that the quotient of an a-T-menable group by a finite normal supgroup is a-T-menable. We conclude with a new proof that the free product of two a-T-menable groups is a-T-menable.

Contents

Acknowledgments

First and foremost, I would like to thank my thesis advisor, Professor Erik Guentner, for introducing me to the world of a-T-menable groups. During my years of study, I benefited from his constant encouragement, extreme patience and guidance. This thesis emerged after our numerous discussions and his constant suggestions for ways to look at things from different perspectives. Most importantly, this work would not have been possible without his confidence in my talent. I have great pleasure in thanking him.

I would like to thank Professor Wayne Smith for agreeing to be on my thesis committe and reading my paper carefully.

I would also like to thank the members of the Graduate Committe for reviewing my thesis.

I would like to thank all Professors in the department who taught me all the mathematics I know now.

My graduate life would not have been as pleasurable without the help of Sue and Shirley and many friends and colleagues. I am happy to thank them. In particular, I would like to thank Dr. Mirjana Jovovic for her shared experience regarding graduate life and beyond.

Lastly, I would like to thank my parents for their enduring encouragement and total support of my decision to study mathematics.

1 Introduction

A discrete group G is a-T-menable (or Haagerup) if it satisfies one of the following conditions:

1. There exists a proper, conditionally negative definite function
$$\psi : G \to \mathbb{R}^+;$$
2. There exists a sequence $(\varphi_n)_{n\geq 1}$ of positive definite functions in $C_0(G)$ such that $\varphi_n(e_G) = 1$, and $\varphi_n \to 1$ pointwise on G.
3. There exists a Hilbert space $\mathcal{H}$ and an affine isometric action α of G on $\mathcal{H}$ which is proper:
$$\forall x \in \mathcal{H}, \forall B \subset \mathcal{H} \text{ bounded}, \#\left\{g \in G : \alpha_g\left(x\right) \in B\right\} < \infty.$$

The above equivalent conditions characterize a class of groups, which is interesting for many reasons. One especially important reason is that a-T-menable groups satisfy the Baum-Connes conjecture. This result was the work of N. Higson and G. Kasparov [HK97]. Some examples of a-T-menable groups include compact groups; Lie groups SO(n,1) and SU(n,1); groups acting properly on trees; Coxeter groups; amenable groups, groups acting on spaces with walls. For a complete list of a-T-menable groups, please consult [CCJ+01, Chapter 1].

2 Preliminaries

In this section, we provide some background that we think is needed to understand this paper. For a more detailed treatment, we recommend that the reader consult [BHV, Appendix C]. We assume that the reader is familiar with concepts such as Hilbert space, linear functional, etc.

Throughout this paper, G is a discrete group and $\mathcal{H}$ is a Hilbert space.

2.1 The group of affine isometries

Definition 1 *An affine isometry is a function $\alpha : \mathcal{H} \to \mathcal{H}$ sending an element $x \in \mathcal{H}$ to $T(x) + b$ where $b \in \mathcal{H}$ and T is a unitary operator of $\mathcal{H}$. Note that this is an isometry of $\mathcal{H}$ because*
$$\|\alpha(x) - \alpha(y)\| = \|T(x) + b - (T(y) + b)\| = \|T(x - y)\| = \|x - y\|$$
for all $x, y \in \mathcal{H}$.

Let α and β be two affine isometries. Then
$$\begin{aligned} \alpha\left(x\right) &= T\left(x\right) + b, \\ \beta\left(y\right) &= S\left(y\right) + c \end{aligned}$$

for some unitary operators T and S and some b and $c \in \mathcal{H}$. The composition

$$\begin{aligned}\alpha \circ \beta(y) &= \alpha(\beta(y)) = \alpha(S(y)+c) \\ &= T(S(y)+c)+b \\ &= T(S(y)) + (T(c)+b)\end{aligned}$$

is an affine isometry of $\mathcal{H}$.

The inverse of α is given by $\alpha^{-1}(y) = T^{-1}(y) - T^{-1}(b)$ because

$$\begin{aligned}\alpha \circ \alpha^{-1}(y) &= \alpha(T^{-1}(y) - T^{-1}(b)) \\ &= T(T^{-1}(y) - T^{-1}(b)) + b \\ &= y,\end{aligned}$$

and

$$\begin{aligned}\alpha^{-1} \circ \alpha(x) &= \alpha^{-1}(T(x)+b) \\ &= T^{-1}(T(x)+b) - T^{-1}(b) \\ &= x,\end{aligned}$$

which also shows that α^{-1} is an affine isometry of $\mathcal{H}$.

The set of affine isometries is therefore a group under composition. We denote this group by $\mathrm{AffIsom}(\mathcal{H})$.

Remark 2 *Denote $U(\mathcal{H})$ the group of unitary operators of $\mathcal{H}$. Then, $\mathit{AffIsom}(\mathcal{H})$ is the semi-direct product $U(\mathcal{H}) \ltimes \mathcal{H}$. Here, $U(\mathcal{H})$ acts on $\mathcal{H}$ in the evident manner.*

Remark 3 *Thinking of $\mathcal{H}$ as an affine space, there is a natural notion of affine map. For real affine spaces, the isometric affine maps are characterized by our Definition 1 by the Mazur-Ulam theorem [MU32]. This is false for complex spaces. For example, complex conjugation is an isometric affine map but is not affine isometry in the sense of Definition 1.*

2.2 Affine isometric actions

Definition 4 *An affine action is a homomorphism $\alpha : G \to \mathit{AffIsom}(\mathcal{H})$.*

Definition 5 *Let $\pi : G \to U(\mathcal{H})$ be a unitary representation of G, ie, a linear representation of G on $\mathcal{H}$ such that π_g is a unitary operator for every $g \in G$. By a cocycle for the map π, we mean a function $b : G \to \mathcal{H}$ satifying the cocycle identity:*

$$b(st) = \pi_s(b(t)) + b(s).$$

Example 6 *Suppose $\xi \in \mathcal{H}$. The function*

$$b : G \to \mathcal{H},\ g \mapsto \pi_g(\xi) - \xi$$

is a cocycle for the unitary map π *because*

$$\begin{aligned} b(st) &= \pi_{st}(\xi) - \xi \\ &= \pi_s \pi_t(\xi) - \xi \\ &= \pi_s(\pi_t(\xi) - \xi) + \pi_s(\xi) - \xi \\ &= \pi_s(b(t)) + b(s). \end{aligned}$$

Remark 7 *Let* e *be the identity element of* G *and* $g \in G$. *One can deduce from Definition 5 that*

$$\begin{array}{ll} (i) & b(e) = 0; \\ (ii) & \pi_g\left(b\left(g^{-1}\right)\right) = -b(g); \\ (iii) & b(e) - b\left(h^{-1}g\right) = \pi_{h^{-1}}(b(g)) - \pi_{h^{-1}}(b(h)). \end{array}$$

Theorem 8 *Given an affine isometric action* α, *one obtains a unitary representation* π *and a cocycle* b *such that*

$$\alpha_g(x) = \pi_g(x) + b(g);$$

and conversely.

Proof. We first prove the forward implication. Viewing $\mathrm{AffIsom}(\mathcal{H})$ as $U(\mathcal{H}) \ltimes \mathcal{H} = \{(T, b)\}$, one sees that $U(\mathcal{H}) \ltimes \mathcal{H}$ projects onto $U(\mathcal{H})$ homomorphically through the map sending (T, b) to T. One can thus recover π from α. To get back the cocycle set $b(g) = \alpha_g(0)$. Indeed, for g and $h \in G$

$$\begin{aligned} b(gh) &= \alpha_{gh}(0) = \alpha_g \alpha_h(0) \\ &= \alpha_g(b(h)) \\ &= \pi_g(b(h)) + b(g) \end{aligned}$$

which shows that the cocycle identity holds.

For the reserve implication, given a unitary representation $\pi : G \to U(\mathcal{H})$ and a cocycle $b : G \to \mathcal{H}$, define

$$\alpha : G \to \mathrm{AffIsom}(\mathcal{H}),\ \alpha_g = \pi_g + b(g).$$

We will verify that α is a homomorphism. For g and $h \in G$, we have

$$\begin{aligned} \alpha_g \alpha_h(x) &= \alpha_g(\pi_h(x) + b(h)) \\ &= \pi_g(\pi_h(x) + b(h)) + b(g) \\ &= \pi_{gh}(x) + \pi_g(b(h)) + b(g) \\ &= \pi_{gh}(x) + b(gh) \\ &= \alpha_{gh}(x). \end{aligned}$$

■

2.3 Positive definite functions

Definition 9 *A function $\varphi : G \to \mathbb{C}$ is positive definite if for all $g_1, ..., g_n \in G$ and all $a_1, ..., a_n \in \mathbb{C}$,*

$$\sum_{i,j} a_i \overline{a}_j \varphi(g_j^{-1} g_i) \geq 0.$$

Lemma 10 *Let π be a unitary representation on a Hilbert space $\mathcal{H}$ and let v be a unit vector. Then,*

$$\phi : G \to \mathbb{C}; \ \phi(g) = \langle \pi_g(v), v \rangle$$

is positive definite.

Proof. Let $g_1, ..., g_n \in G$ and $a_1, ..., a_n \in \mathbb{C}$. Then

$$\begin{aligned}
\sum_{i,j} a_i \overline{a}_j \phi(g_j^{-1} g_i) &= \sum_{i,j} a_i \overline{a}_j \left\langle \pi_{g_j^{-1} g_i}(v), v \right\rangle \\
&= \sum_{i,j} a_i \overline{a}_j \left\langle \pi_{g_j^{-1}} \pi_{g_i}(v), v \right\rangle \\
&= \sum_{i,j} a_i \overline{a}_j \left\langle \pi_{g_i}(v), \pi_{g_j}(v) \right\rangle \\
&= \sum_{i,j} \left\langle a_i \pi_{g_i}(v), a_j \pi_{g_j}(v) \right\rangle \\
&= \left\langle \sum_i a_i \pi_{g_i}(v), \sum_j a_j \pi_{g_j}(v) \right\rangle \\
&= \left\| \sum_i a_i \pi_{g_i}(v) \right\|^2 \\
&\geq 0.
\end{aligned}$$

■

Definition 11 *A positive definite kernel on a group G is a function $\Phi : G \times G \longrightarrow \mathbb{C}$ such that for all $g_1, ..., g_n \in X$ and all $a_1, ..., a_n \in \mathbb{C}$,*

$$\sum_{i,j} a_i \overline{a}_j \Phi(g_i, g_j) \geq 0.$$

Remark 12 *A function φ on G is positive definite if and only if the kernel $(g, h) \longmapsto \Phi(g, h) = \varphi\left(h^{-1} g\right)$ is positive definite.*

The next result is a version of the Gelfand-Naimark-Segal (GNS) construction.

Theorem 13 *Let Φ be kernel on G. Then Φ is a positive definite kernel if and only if there is a Hilbert space $\mathcal{H}$ and a mapping $v : G \longrightarrow \mathcal{H}$ such that*

$$\Phi(g,h) = \langle v(g), v(h) \rangle, \ \forall g,h \in G.$$

Proof. First, we will prove the forward implication.

Let $V = \{\xi : G \to \mathbb{C} : \xi \text{ is a finitely supported function}\}$. Then V is a vector space. For $\xi_1, \xi_2 \in V$, define

$$\langle \xi_1, \xi_2 \rangle = \sum_{g,h} \xi_1(g) \overline{\xi_2(h)} \Phi(g,h).$$

It is not hard to see that this is linear in the first variable and anti-linear in the second variable. For anti-symmetric, we note that one needs $\overline{\Phi(h,g)} = \Phi(g,h)$ but this follows directly from the definition. We will verify the positive semidefiniteness condition. For $\xi \in \mathcal{H}$,

$$\begin{aligned} \langle \xi, \xi \rangle &= \sum_{g,h} \xi(g) \overline{\xi(h)} \Phi(g,h) \\ &= \sum_{i,j} \xi(g_i) \overline{\xi(g_j)} \Phi(g_i, g_j) \geq 0, \end{aligned}$$

where $\{g_1, ..., g_n\} \subset G$ is the support of ξ.

Let $\mathcal{H}$ be the Hilbert space obtained after separating and completing V and define $v : G \to \mathcal{H}$ by $v(g) = \delta_g$, the Dirac function at g. Then:

$$\begin{aligned} \langle v(g), v(h) \rangle &= \sum_{a,b} v(g)(a) \overline{v(h)(b)} \Phi(a,b) \\ &= \sum_{a,b} \delta_g(a) \overline{\delta_h(b)} \Phi(a,b) \\ &= \delta_g(g) \overline{\delta_h(h)} \Phi(g,h) \\ &= \Phi(g,h). \end{aligned}$$

The reverse direction is similar to Lemma 10. We verify it here. Let $g_1, ..., g_n \in G$ and $a_1, ..., a_n \in \mathbb{C}$. Then:

$$\begin{aligned} \sum_{i,j} a_i \overline{a}_j \Phi(g_i, g_j) &= \sum_{i,j} \langle a_i v(g_i), a_j v_j(g_j) \rangle \\ &= \sum_{i} \|a_i v(g_i)\|^2 \geq 0. \end{aligned}$$

■

Proposition 14 *If Φ, Ψ are two positive definite kernels, then so is $\Phi\Psi$.*

Proof. By Theorem 13, there are Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and maps $v : G \to \mathcal{H}$ and $w : G \to \mathcal{K}$ such that $\Phi(g,h) = \langle v(g), v(h) \rangle$ and $\Psi(g,h) = \langle w(g), w(h) \rangle$. Define $u : G \to \mathcal{H} \otimes \mathcal{K}$ by $u(g) = v(g) \otimes w(g)$. Then

$$\begin{aligned} \Phi(g,h)\Psi(g,h) &= \langle v(g), v(h) \rangle \langle w(g), w(h) \rangle \\ &= \langle v(g) \otimes w(g), v(h) \otimes w(h) \rangle \\ &= \langle u(g), u(h) \rangle \end{aligned}$$

is a positive definite kernel, by Theorem 13. ■

2.4 Conditionally negative type functions

Definition 15 *A function $\psi : G \to \mathbb{R}$ is conditionally negative definite if and only if $\psi(g^{-1}) = \psi(g)$ for all $g \in G$, and for all $g_1, ..., g_n \in G$ and all $a_1, ..., a_n \in \mathbb{R}$ with $\sum a_i = 0$,*

$$(\bigstar) \sum_{i,j} a_i a_j \psi(g_j^{-1} g_i) \leq 0.$$

Definition 16 *Let e denote the identity element of G. A conditionally negative definite function ψ is normalized provided $\psi(e) = 0$.*

Remark 17 *Note that when $\psi(e) = 0$, $\psi(g) \geq 0\ \forall g$. To verify this, put $a_1 = 1,\ a_2 = -1,\ g_1 = g,\ g_2 = e$ into ($\bigstar$) and use the hypothesis $\psi(g^{-1}) = \psi(g)$ for all $g \in G$.*

Example 18 *Let α be an affine isometric action of G on a real Hilbert space $\mathcal{H}$. For any $\xi \in \mathcal{H}$, the function*

$$\psi : G \to \mathbb{R},\ g \mapsto \|\alpha_g(\xi) - \xi\|^2$$

is conditionally negative definite.

Remark 19 *One can put $\xi = 0$ in the above formula for ψ and obtain $\psi(g) = \|\alpha_g(0)\|^2 = \|b(g)\|^2$ where b is the cocycle corresponding to the unitary representation π recovered from α.*

Below is Schoenberg's Theorem.

Theorem 20 *Let G be a group and let $\psi : G \to \mathbb{R}$. Then ψ is a normalized, conditionally negative definite function if and only $e^{-t\psi}$ is positive definite for all $t \geq 0$.*

Remark 21 *Here e is used to mean the exponential function. We trust that the reader is not confused between the two notions of e.*

Before proving this theorem, we need some definitions and lemmas.

Definition 22 *Let X be a set. A conditionally negative type kernel on X is a function $\Psi : X \times X \to \mathbb{R}$ such that:*

1. *$\Psi(y, x) = \Psi(x, y)$ for all $x, y \in X$;*
2. *For all $x_1, ..., x_n \in X$ and all $c_1, ..., c_n \in \mathbb{R}$ with $\sum_i c_i = 0$ we have $\sum_{i,j} c_i c_j \Psi(x_i, x_j) \leq 0$.*

Definition 23 *A conditionally negative type kernel ψ is called normalized provided $\Psi(x, x) = 0$ for all $x \in X$.*

Example 24 *The kernel $\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by $\Psi(\xi, \eta) = \|\xi - \eta\|^2$ is conditionally of negative type.*

Remark 25 *A function $\psi : G \to \mathbb{R}$ is conditionally of negative type if the kernel Ψ on G, defined by $\Psi(g, h) = \psi(h^{-1}g)$ is conditionally of negative type.*

Remark 26 *One can also work with $\mathbb{C}-$Hilbert spaces by replacing real scalars in Definition 15 by complex scalars and replacing ($\bigstar$) by*

$$\sum_{i,j} a_i \overline{a}_j \psi(g_j^{-1} g_i) \leq 0.$$

Lemma 27 *A normalized conditionally negative type kernel on X with real scalars is also a conditionally negative type kernel on X with complex scalars.*

Proof. We have to show that if Ψ is conditionally negative definite in the sense of Definition 15 (with real scalars) then it is conditionally negative definite definite in the sense of Remark 26 (with complex scalars). Let $x_1, ..., x_n \in X$ and all $c_1, ..., c_n \in \mathbb{C}$ with $\sum_j c_j = 0$. Write $c_j = a_j + ib_j$ where $a_j, b_j \in \mathbb{R}$ then $\sum_j a_j = 0$ and $\sum_j b_j = 0$. Now

$$\begin{aligned}
\sum_{j,k} c_j \overline{c}_k \Psi(x_j, x_k) &= \sum_{j,k} (a_j + ib_j) \overline{(a_k + ib_k)} \Psi(x_j, x_k) \\
&= \sum_{j,k} (a_j + ib_j)(a_k - ib_k) \Psi(x_j, x_k) \\
&= \sum_{j,k} (a_j a_k + b_j b_k + i(b_j a_k - a_j b_k)) \Psi(x_j, x_k) \\
&= \sum_{j,k} (a_j a_k + b_j b_k) \Psi(x_j, x_k) \\
&= \sum_{j,k} (a_j a_k) \Psi(x_j, x_k) + \sum_{j,k} (b_j b_k) \Psi(x_j, x_k) \\
&\leq 0.
\end{aligned}$$

■

Lemma 28 *Let $\psi : G \to [0,\infty)$ be a function. Then ψ is a normalized conditionally negative definite function if and only if there exists a Hilbert space $\mathcal{H}$ and a map $b : G \to \mathcal{H}$ such that $\psi\left(h^{-1}g\right) = \|b(g) - b(h)\|^2$.*

Proof. First, we will prove the forward implication. Let $\psi : G \to [0,\infty)$ be a normalized conditionally negative definite function. Let

$$\mathcal{H} = \left\{ \xi : G \to \mathbb{C} : \sum_{g\in G} \xi(g) = 0, |\{g : \xi(g) \neq 0\}| < \infty \right\}.$$

Then $\mathcal{H} \neq \emptyset$ because the zero function is in it. Then one can verify that $\mathcal{H}$ is a vector space. For a pair of vectors $\xi, \eta \in \mathcal{H}$, we define :

$$\langle \xi, \eta \rangle = -\frac{1}{2} \sum_{g,h} \xi(g)\, \overline{\eta(h)} \psi\left(h^{-1}g\right).$$

This is a positive semidefinite inner product because it is linear in the first varible, conjugate symmetric and positive semidefinite. We verify these assertions below.

i. Linearity in the first variable: $\langle a\xi, \eta\rangle = a\langle \xi, \eta\rangle$ is obvious and

$$\begin{aligned}
\langle \xi_1 + \xi_2, \eta \rangle &= -\frac{1}{2}\sum_{g,h} (\xi_1 + \xi_2)(g)\, \overline{\eta(h)}\psi\left(h^{-1}g\right) \\
&= -\frac{1}{2}\sum_{g,h} (\xi_1(g) + \xi_2(g))\, \overline{\eta(h)}\psi\left(h^{-1}g\right) \\
&= -\frac{1}{2}\sum_{g,h} \xi_1(g)\, \overline{\eta(h)}\psi\left(h^{-1}g\right) - \frac{1}{2}\sum_{g,h} \xi_2(g)\, \overline{\eta(h)}\psi\left(h^{-1}g\right) \\
&= \langle \xi_1, \eta \rangle + \langle \xi_2, \eta \rangle .
\end{aligned}$$

ii. Symmetry:

$$\begin{aligned}
\overline{\langle \eta, \xi \rangle} &= \overline{-\frac{1}{2}\sum_{g,h} \eta(g)\, \overline{\xi(h)}\psi\left(h^{-1}g\right)} \\
&= -\frac{1}{2}\sum_{g,h} \overline{\eta(g)}\xi(h)\psi\left(\left(h^{-1}g\right)^{-1}\right) \\
&= -\frac{1}{2}\sum_{g,h} \xi(h)\overline{\eta(g)}\psi\left(g^{-1}h\right) \\
&= \langle \xi, \eta \rangle .
\end{aligned}$$

iii. Positive semidefiniteness:

$$\begin{aligned}\langle \xi, \xi \rangle &= -\frac{1}{2}\sum_{g,h} \xi(g)\overline{\xi(h)}\psi\left(h^{-1}g\right) \\ &= -\frac{1}{2}\sum_{i,j} \xi(g_i)\overline{\xi(g_j)}\psi\left(g_j^{-1}g_i\right) \\ &= -\frac{1}{2}\sum_{i,j} a_i\overline{a}_j\psi\left(g_j^{-1}g_i\right) \geq 0,\end{aligned}$$

where in the second equality, we use the fact that the function ξ is finitely supported and $\{g_1, ...g_n\} \subset G$ is the support of ξ.

Now, let $N = \{\xi \in \mathcal{H} : \langle \xi, \xi \rangle = 0\}$. Then N is a vector subspace of $\mathcal{H}$ and form the quotient space $\mathcal{H}/N$. Equip $\mathcal{H}/N$ with the inner product

$$\langle \xi + N, \eta + N \rangle = \langle \xi, \eta \rangle .$$

The completion of $\mathcal{H}/N$ is the Hilbert space we want.

For $g \in G$, we define a function

$$b : G \to \mathcal{H} \text{ by } b(g) = \delta_g - \delta_e$$

where δ is the Dirac function and e is the identity element of G. We claim that $\psi\left(h^{-1}g\right) = \|b(g) - b(h)\|^2$ for $g, h \in G$. Indeed,

$$\begin{aligned}\|b(g) - b(h)\|^2 &= \|\delta_g - \delta_e - (\delta_h - \delta_e)\|^2 \\ &= \|\delta_g - \delta_h\|^2 \\ &= \langle \delta_g - \delta_h, \delta_g - \delta_h \rangle \\ &= -\frac{1}{2}\sum_{k,l} (\delta_g - \delta_h)(k)\overline{(\delta_g - \delta_h)(l)}\psi\left(l^{-1}k\right) \\ &= -\frac{1}{2}\sum_{k,l} \delta_g(k)\overline{\delta_g(l)}\psi\left(l^{-1}k\right) - \frac{1}{2}\sum_{k,l} \delta_h(k)\overline{\delta_h(l)}\psi\left(l^{-1}k\right) \\ &\quad - \left(-\frac{1}{2}\sum_{k,l} \delta_g(k)\overline{\delta_h(l)}\psi\left(l^{-1}k\right)\right) - \left(-\frac{1}{2}\sum_{k,l} \delta_h(k)\overline{\delta_g(l)}\psi\left(\left(l^{-1}k\right)^{-1}\right)\right) \\ &= 0 + 0 + \frac{1}{2}\sum_{k,l} \delta_g(k)\overline{\delta_h(l)}\psi\left(l^{-1}k\right) + \frac{1}{2}\sum_{k,l} \delta_g(l)\overline{\delta_h(k)}\psi\left(k^{-1}l\right) \\ &= \frac{1}{2}\psi\left(h^{-1}g\right) + \frac{1}{2}\psi\left(h^{-1}g\right) = \psi\left(h^{-1}g\right),\end{aligned}$$

where in the sixth equality we use the fact that ψ is normalized.

We prove the reverse implication here. Let $a_1, ..., a_n \in \mathbb{C}$ with $\sum a_i = 0$

and $g_1, ..., g_n \in G$, then:

$$\begin{aligned}
\sum_{i,j} a_i \overline{a}_j \psi\left(g_j^{-1} g_i\right) &= \sum_{i,j} a_i \overline{a}_j \left\| b\left(g_i\right) - b\left(g_j\right) \right\|^2 \\
&= \sum_{i,j} a_i \overline{a}_j \left\langle b\left(g_i\right) - b\left(g_j\right), b\left(g_i\right) - b\left(g_j\right) \right\rangle \\
&= \sum_{i,j} a_i \overline{a}_j [\left\langle b\left(g_i\right), b\left(g_i\right) \right\rangle - \left\langle b\left(g_i\right), b\left(g_j\right) \right\rangle - \left\langle b\left(g_j\right), b\left(g_i\right) \right\rangle + \left\langle b\left(g_j\right), b\left(g_j\right) \right\rangle] \\
&= \sum_j \overline{a}_j \left(\sum_i a_i \left\langle b\left(g_i\right), b\left(g_i\right) \right\rangle \right) - \sum_{i,j} a_i \overline{a}_j \left\langle b\left(g_i\right), b\left(g_j\right) \right\rangle \\
&\quad + \sum_i a_i \left(\sum_j \overline{a}_j \left\langle b\left(g_j\right), b\left(g_j\right) \right\rangle \right) - \sum_{i,j} a_i \overline{a}_j \overline{\left\langle b\left(g_i\right), b\left(g_j\right) \right\rangle}] \qquad (1) \\
&= -\left\| \sum_i a_i b\left(g_i\right) \right\|^2 - \left\| \sum_i \overline{a}_i b\left(g_i\right) \right\|^2 \leq 0.
\end{aligned}$$

■

Lemma 29 *Suppose $\Psi : G \times G \to [0, \infty)$. Then Ψ is a normalized conditionally negative type kernel if and only if there exists a Hilbert space $\mathcal{H}$ and a map $b : G \to \mathcal{H}$ such that $\Psi(g, h) = \|b(g) - b(h)\|^2$.*

Proof. The proof of this is similar to the one in the previous lemma, but with a slight modification. ■

Lemma 30 *Let Ψ be a conditionally negative type kernel on X and fix $x_0 \in X$. Define $\Phi : X \times X \to \mathbb{R}$ by $\Phi(x, y) = \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y)$. Then Φ is a positive type kernel.*

Proof. By the previous lemma, there is a Hilbert space $\mathcal{H}$ and a normalized map $b : G \to \mathcal{H}$ such that

$$\begin{aligned}
\Psi(x, x_0) &= \|b(x) - b(x_0)\|^2 \\
\Psi(y, x_0) &= \|b(y) - b(x_0)\|^2 \\
\Psi(x, y) &= \|b(x) - b(y)\|^2
\end{aligned}$$

for all $x, y \in X$. We have:

$$\begin{aligned}
\Phi(x, y) &= \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y) \\
&= \|b(x) - b(x_0)\|^2 + \|b(y) - b(x_0)\|^2 - \|b(x) - b(y)\|^2 \\
&= 2\langle b(x) - b(x_0), b(y) - b(x_0) \rangle,
\end{aligned}$$

which shows that Φ is a positive definite kernel by Theorem 13. ■

Now, we are ready to prove Schoenberg's Theorem. Although it is an if and only if statement, we only need the forward direction for this paper. The proof of this implication is provided below.

Proof. Assume Ψ is a normalized, conditionally negative type kernel. We want to prove that $e^{-t\Psi}$ is positive definite for all $t \geq 0$. We note if Ψ is a normalized, conditionally negative type kernel then $t\Psi$ is also a Ψ is a normalized, conditionally negative type kernel. Therefore, it is enough to prove the theorem for the case $t = 1$.

Fix $x_0 \in X$. By the previous lemma:

$$\Phi(x, y) = \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y)$$

is a positive type kernel. Therefore

$$-\Psi(x, y) = \Phi(x, y) - \Psi(x, x_0) - \Psi(y, x_0)$$

and so

$$e^{-\Psi(x,y)} = e^{\Phi(x,y)} e^{-\Psi(x,x_0)} e^{-\Psi(y,x_0)}.$$

We will prove that $e^{\Phi(x,y)}$and $e^{-\Psi(x,x_0)} e^{-\Psi(y,x_0)}$ are of positive type and so $e^{-\Psi(x,y)}$ is of positive type by Proposition 14. Indeed, since Φ is of positive type, $\Phi(x, y)^n$ is of positive type for any $n \geq 0$ and so $e^{\Phi(x,y)}$ is positive definite as it is the pointwise limit of positive definite kernels.

For all $x_1, ..., x_n \in X$ and $c_1, ..., c_n \in \mathbb{C}$

$$\sum_{i,j} c_i \bar{c}_j e^{-\Psi(x_i,x_0)} e^{-\Psi(x_j,x_0)} = \left| \sum_i c_i e^{-\Psi(x_i,x_0)} \right|^2 \geq 0,$$

which shows that the kernel $(x, y) \longmapsto e^{-\Psi(x,x_0)} e^{-\Psi(y,x_0)}$ is of positive type. ■

3 A-T-menable groups

In this section, we will first state three conditions charaterizing a-T-menability of groups and prove that they are equivalent. We will then work out explicitly an example illustrating that the free group on 2 generators is a-T-menable. We will describe a connection between amenability and a-T-menability by proving that the infinite cyclic group is amenble and therefore is a-T-menable.

3.1 Characterization

We state several conditions for a-T-menability in the introduction. One therefore may wonder how these conditions are related. We provide proof of the equivalences below.

Proof. We shall prove the equivalence of (1) and (2) and also the equivalence of (3) and (1).

(1 $\Longrightarrow$ 2) If ψ is a proper, conditionally negative definite function in (1) and $\psi(e) \neq 0$, we can define $\psi' = \psi - \psi(e)$ then ψ' is a proper, normalized, conditionally definite function. We thus can assume that ψ is normalized. By Schonberg's theorem, $e^{-t^2\psi}$ is positive definite.

Let $n \in \mathbb{N}$, then $(\varphi_n(g))_{n\geq 1} = \left(e^{-\frac{1}{n}\psi(g)}\right)_{n\geq 1}$ is a sequence of positive definite functions on G. We see that $\varphi_n(e) = 1$ because $\varphi_n(e) = e^{-\frac{1}{n}\psi(e)} = e^{-\frac{1}{n}0} = e^0 = 1$ and from properness of ψ, $\varphi_n(g) \in C_0(G)$.

It remains to check that $(\varphi_n)_{n\geq 1}$ converges to 1 pointwise on G. This is obvious since

$$\lim_{n\longrightarrow\infty} e^{-\frac{1}{n}\psi(g)} = e^{\lim\limits_{n\longrightarrow\infty}\left(-\frac{1}{n}\psi(g)\right)} = e^0 = 1.$$

(2 $\Longrightarrow$ 1) Assume that there is a sequence $(\varphi_n)\ _{n\geq 1}$ of positive definite functions such that $\varphi_n(e) = 1,\ \varphi_n(g) \to 1$ for all $g \in G$ and $\varphi_n \in C_0(G)$.

1. We may always assume $0 \leq \varphi_n(g) \leq 1$ since if φ is positive definite then $\overline{\varphi}$ is positive definite and so is $|\varphi|^2$ and $|\varphi(g)|^2 \leq |\varphi(e)|^2 = 1$ for all $g \in G$ [BHV, Proposition C.4.2].

2. We may find sequences

   $$\begin{array}{lcl} K_1 & \subset & K_2 \subset ... \subset K_k \subset ... \subset G \text{ with } \cup K_k = G \text{ and } K_k \text{ is finite for each } k \text{ and} \\ n_1 & < & n_2 < ... < n_k < ... \end{array}$$

   such that

   $$\begin{array}{rcl} \left|1 - \varphi_{n_k}\right| & \leq & \dfrac{1}{2^k} \text{ on } K_k, \\ \varphi_{n_k} & < & \dfrac{1}{2} \text{ on } K_{k+1}^C. \end{array}$$

3. Then: $\sum\left(1 - \varphi_{n_k}\right)$ is

   (i) convergent for all $g \in G$: Let $g \in G$. Then either $g \in K_k$ for some k, hence also then $g \in K_i,\ \forall i \geq k$. Therefore

   $$\begin{array}{rcl} \displaystyle\sum_{j=1}^{\infty}\left(1 - \varphi_{n_j}(g)\right) & = & \displaystyle\sum_{j=1}^{k-1}\left(1 - \varphi_{n_j}(g)\right) + \sum_{j=k}^{\infty}\left(1 - \varphi_{n_j}(g)\right) \\ & \leq & \displaystyle\sum_{j=1}^{k-1}\left(1 - \varphi_{n_j}(g)\right) + \sum_{n\geq n_k}\frac{1}{2^n} \\ & < & \infty. \end{array}$$

   (ii) negative type: for each n_k if φ_{n_k} is positive type then $-\varphi_{n_k}$ is negative type and so is $1 - \varphi_{n_k}$. Since $\sum\left(1 - \varphi_{n_k}\right)$ is the convergent sum of negative type functions, it is a negative type function.

(iii) proper: Let $g \in K^C_{n_{N+1}}$. Then $g \in K^C_{n_j}$ for all $j \leq N+1$. Hence, $\varphi_{n_j}(g) < \frac{1}{2}$ for all $j \leq N$ and so $\left(1 - \varphi_{n_j}(g)\right) \geq \frac{1}{2}$ for all $j \leq N$. Thus,

$$\sum \left(1 - \varphi_{n_k}(g)\right) \geq \sum_{1}^{N} \frac{1}{2} = \frac{N}{2},$$

which $\to \infty$ when N large.

(1 $\Longrightarrow$ 3) Suppose that we have a proper, conditionally negative definite function ψ. As in the proof of (1) $\Longrightarrow$ (2), we may assume ψ is normalized. By lemma 28, there exists a Hilbert space $\mathcal{H}$ and a function

$$b : G \to \mathcal{H} \text{ such that } b(g) = \delta_g - \delta_e,$$

where

$$b(g)(h) = \delta_g(h) - \delta_e(h) = \begin{cases} +1, & h = g \neq e \\ -1, & h = e \neq g \\ 0, & \text{else} \end{cases}$$

for $h \in G$.

Let $\xi \in \mathcal{H}$. We define a unitary representation of G on the Hilbert space from the lemma as follows: for $g \in G$, let $\pi_g : \mathcal{H} \to \mathcal{H}$ be defined by $\pi_g \xi(k) = \xi(g^{-1}k)$. We verify that π is a unitary representation and b satisfies the cocycle identity below:

(i) Unitary:

$$\begin{aligned}
\langle \pi_g \xi, \pi_g \eta \rangle &= -\frac{1}{2} \sum_{a,b} \pi_g \xi(a) \overline{\pi_g \eta(b)} \psi\left(b^{-1}a\right) \\
&= -\frac{1}{2} \sum_{a,b} \xi(g^{-1}a) \overline{\eta(g^{-1}b)} \psi\left(b^{-1}a\right) \\
&= -\frac{1}{2} \sum_{a,b} \xi(g^{-1}a) \overline{\eta(g^{-1}b)} \psi\left(b^{-1}gg^{-1}a\right) \\
&= -\frac{1}{2} \sum_{g^{-1}a, g^{-1}b} \xi(g^{-1}a) \overline{\eta(g^{-1}b)} \psi\left(\left(g^{-1}b\right)^{-1}\left(g^{-1}a\right)\right) \\
&= -\frac{1}{2} \sum_{a',b'} \xi(a') \overline{\eta(b')} \psi\left((b')^{-1} a'\right) \\
&= \langle \xi, \eta \rangle .
\end{aligned}$$

(ii) Representation: for $k \in G$

$$\pi_{gh}\xi(k) = \xi((gh)^{-1}k) = \xi(h^{-1}g^{-1}k) = (\pi_h \xi)(g^{-1}k) = \pi_g \pi_h \xi(k).$$

(iii) The cocycle identity for b :

$$\begin{aligned}\pi_s(b(t))(h)+b(s)(h) &= b(t)\left(s^{-1}h\right)+b(s)(h)\\ &= \delta_t\left(s^{-1}h\right)-\delta_e\left(s^{-1}h\right)+\left(\delta_s(h)-\delta_e(h)\right)\\ &= \delta_{st}(h)-\delta_s(h)+\left(\delta_s(h)-\delta_e(h)\right)\\ &= \delta_{st}(h)-\delta_e(h)\\ &= b(st),\end{aligned}$$

where the third equality is true since

$$\begin{aligned}\delta_t\left(s^{-1}h\right)-\delta_e\left(s^{-1}h\right) &= \begin{cases}+1, & s^{-1}h=t\neq e\\ -1, & s^{-1}h=e\neq t\\ 0, & \text{else}\end{cases}\\ &= \begin{cases}+1, & h=st\neq s\\ -1, & h=s\neq st\\ 0, & \text{else}\end{cases}\\ &= \delta_{st}(h)-\delta_s(h).\end{aligned}$$

By Theorem 8, there is an affine isometric action

$$\alpha_g : G\rightarrow \text{ AffIsom}(\mathcal{H}),\ \alpha_g=\pi_g+b(g).$$

It remains to show that α is proper. Observe that

$$\|b(g)\|^2=\psi(g)$$

by Lemma 28. It follows that α_g is proper as $\psi(g)$ is.

(3 $\Longrightarrow$ 1) If α is an affine isometric action of G then by example 18 and remark 19, there is a cocycle b corresponding to the unitary representation recovered from α. Let $g\in G$ and define $\psi(g)=\|b(g)\|^2$. We assert that $\psi : G\rightarrow[0,\infty)$ is the conditionally negative definite function satisfying (1). Note that

$$\psi\left(g^{-1}\right)=\left\|b(g^{-1})\right\|^2=\|-\pi_g(b(g))\|^2=\|b(g)\|^2=\psi(g)$$

and so

$$\begin{aligned}\psi(h^{-1}g) &= \left\|b\left(h^{-1}g\right)\right\|^2\\ &= \left\|b(e)-b\left(h^{-1}g\right)\right\|^2\\ &= \|b(g)-b(h)\|^2\end{aligned}$$

where the third equality follows from remark 7. Thus, ψ is a conditionally negative definite function by lemma 28.

As in the proof of $(1)\Rightarrow(3)$, we see that ψ is proper if and only if α is. ■

Definition 31 *A group is a-T-menable if it satisfies one of the conditions in the previous theorem.*

3.2 The free group on two generators $\mathcal{F}_2$

A standard example of an a-T-menable group is the free group. Associating to a free group its Cayley Graph, one can construct a Hilbert space as the space of functions on the set of oriented edges. One can then define an affine isometric action using this Hilbert space.

In this section, we use the free group on two generators $\mathcal{F}_2$ as a leading example. We will study the Cayley graph of $\mathcal{F}_2$ and present two proofs that $\mathcal{F}_2$ is a-T-menable.

What follows are some basic definitions of graphs, vertices, edges, etc...

Definition 32 *A graph Γ consists of a pair (V, E) where V is the set of vertices and $E \subset V \times V$ is just a subset for which*

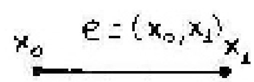
1. $(x, y) \in E \Rightarrow (y, x) \in E$,
2. $(x, x) \notin E$.

Intuitively, (x, y) is the edge that points from x to y;

We use the notation $\overline{e} = (y, x)$ when $e = (x, y)$.

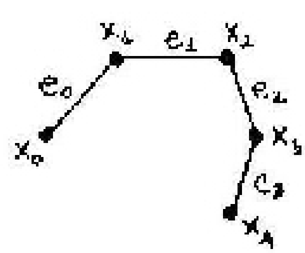
Diagram: We use a diagram to represent the graph, where points correspond to vertices and lines connecting points correspond to edges.

Example 33 *The graph having two vertices x_0 and x_1 and two edges e and $\overline{e}$ is represented by the following diagram*



.

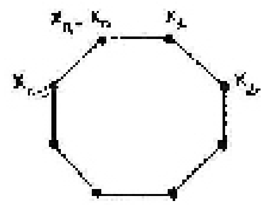
Path: Let $u, v \in V$. Then $[x_0,\ x_1, ..., x_n]$ is a path connecting u and v if $u = x_0,\ v = x_n,\ [x_i, x_{i+1}] \in E$ for $0 \leq i < n$. The length of this path is n.



Let us rewrite the path connecting u and v as $e_0 e_1 ... e_n$ where $e_i = [x_i, x_{i+1}]$. A pair $e_j \overline{e}_j$ in this path is called a backtrack. Thus, we can construct a path of length $n-2$ from u to v by deleting these two terms in the sequence. By induction, there is a path from u to v without backtracking.

Connected graph; a graph Γ is connected if there is a path between any two vertices of Γ.

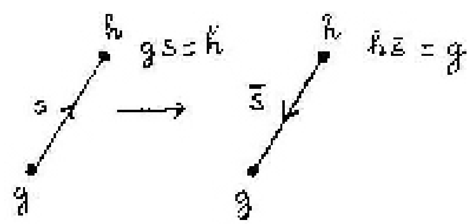
Circuit: A path $[x_0,\ x_1, ..., x_n]$ is a circuit if $x_0 = x_n,\ n \geq 3$.



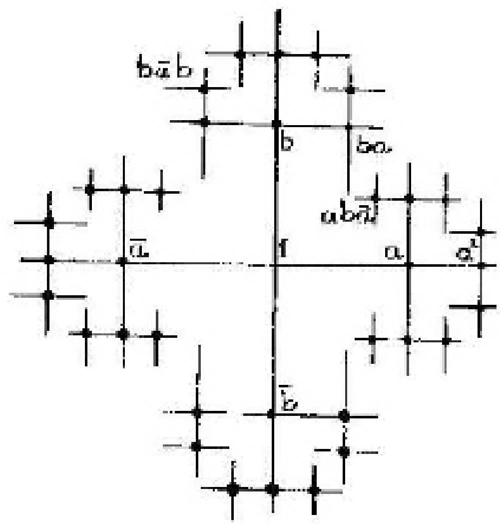
Definition 34 *A tree is a connected non-empty graph without circuits.*

The Cayley graph $\Gamma(G, S)$

Let G be a group and S a symmetric generating set of G, ie, if $s \in S$ then $\overline{s} \in S$, and $e_G \notin S$. We denote $\Gamma(G, S) = (V, E)$ the Cayley graph having G as its set of vertices V and E is the set of pairs $(g, h) \in V \times V$ such that there is $s \in S$ with $gs = h$. See the diagram.



The Cayley graph of $\mathcal{F}_2$: Let $S = \{a,\ b,\ \overline{a},\ \overline{b}\}$ denote the set of generators of $\mathcal{F}_2$ then $V = \mathcal{F}_2$ and $E = \{(g, h) : \exists s \in S \text{ such that } gs = h\}$. The Cayley graph of $\mathcal{F}_2$ is represented by the following diagram



.

Theorem 35 *$\mathcal{F}_2$ is a-T-menable.*

Proof. In order to show that $\mathcal{F}_2$ is a-T-menable, we will need a unitary representation $\pi : G \to U(\mathcal{H})$ and a cocycle $b : G \to \mathcal{H}$. Let $\mathcal{H} = l^2(E) = \overline{\text{span}}$ $\{\delta_e, e \in E\}$.

Let

$$\begin{aligned} \pi &: G \to U(\mathcal{H}), \\ \pi_h\left(\delta_{(g,g')}\right) &= \delta_{h(g,g')} = \delta_{(hg,hg')}. \end{aligned}$$

Observe that if (g, g') is an edge then so is (hg, hg'). We verify that π is a unitary representation of G. For $h, k \in G$

$$\begin{aligned} \pi_{hk}\left(\delta_{(g,g')}\right) &= \delta_{hk(g,g')} = \delta_{(hkg,hkg')} = \delta_{h(kg,kg')} \\ &= \pi_h\left(\delta_{(kg,kg')}\right) = \pi_h\left(\delta_{k(g,g')}\right) = \pi_h\pi_k\left(\delta_{(g,g')}\right) \end{aligned}$$

and

$$\begin{aligned} \left\langle \pi_h\left(\delta_{(g_1,g_1')}\right), \pi_h\left(\delta_{(g_2,g_2')}\right)\right\rangle &= \left\langle \delta_{h(g_1,g_1')}, \delta_{h(g_2,g_2')}\right\rangle \\ &= \left\langle \delta_{(hg_1,hg_1')}, \delta_{(hg_2,hg_2')}\right\rangle \\ &= \left\langle \delta_{(g_1,g_1')}, \delta_{(g_2,g_2')}\right\rangle \end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
\left\langle \delta_{\left(hg_1,hg_1'\right)}, \delta_{\left(hg_2,hg_2'\right)} \right\rangle &= 1 \\
&\Leftrightarrow \left(hg_1, hg_1'\right) = \left(hg_2, hg_2'\right) \\
&\Leftrightarrow h\left(g_1, g_1'\right) = h\left(g_2, g_2'\right) \\
&\Leftrightarrow (g, s) = (g', s') \\
&\Leftrightarrow \left\langle \delta_{(g,s)}, \delta_{(g',s')} \right\rangle = 1.
\end{aligned}$$

Note that the Cayley graph of $\mathcal{F}_2$ has an origin, call it 1. An element g in the set of vertices of $\mathcal{F}_2$ is connected to 1 by some edges $e_1, ...e_n$. We define the cocycle $b : G \to H,\ g \mapsto b(g) = \delta_{e_1} + ... + \delta_{e_n} - \delta_{\overline{e}_1} - ... - \delta_{\overline{e}_n}$. Thus,

$$b(g)(e) = \begin{cases} +1, & e \in [1, g] \\ -1, & e \in [g, 1] \\ 0, & \text{else}, \end{cases}$$

where $[a, b]$ denote the path consisting of edges pointing from a to b. We note that by an edge $[x, y]$ pointing towards a particular g, we mean that y is in the path without backtrack from x to g. If y is not in the path without backtrack from x to g then we say that $[x, y]$ points away from g.Thus, $b(g)$ a well-defined map in $l^2(E)$. We will verify that b satisfies the cocycle identity. Note that we can rewrite the equation of the cocycle as

$$b(g) = \delta_{e_1} + ... + \delta_{e_n} - \delta_{\overline{e}_1} - ... - \delta_{\overline{e}_n} = \chi_{[1,g]} - \chi_{[g,1]},$$

where $\chi_{[1,g]}$ is the characteristic function of the oriented interval $[1, g]$. Let $g, h \in G$. Then

$$\begin{aligned}
\pi_g(b(h)) + b(g) &= \pi_g\left(\chi_{[1,h]} - \chi_{[h,1]}\right) + \chi_{[1,g]} - \chi_{[g,1]} \\
&= \left(\chi_{[g,gh]} - \chi_{[gh,g]}\right) + \chi_{[1,g]} - \chi_{[g,1]} \\
&= \left(\chi_{[1,g]} + \chi_{[g,gh]}\right) - \left(\chi_{[gh,g]} + \chi_{[g,1]}\right) \\
&= \chi_{[1,gh]} - \chi_{[gh,1]} \\
&= b(gh)
\end{aligned}$$

where the fourth equality follows from the lemma below.

Now $\alpha_g = \pi_g + b(g)$ is an affine isometric action of G on $l^2(E)$. It remains to show that α_g is proper. Let n denote the number of edges between 1 and g. Since

$$\|b(g)\|^2 = 2n \to \infty \text{ as } n \to \infty,$$

$b(g)$ is proper function and so is α_g. ■

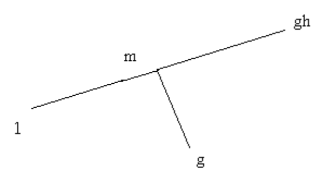
Lemma 36 *In this lemma, we will prove that for $g, h \in \mathcal{F}_2$*

$$\chi_{[1,g]} + \chi_{[g,gh]} - \chi_{[gh,g]} - \chi_{[g,1]} = \chi_{[1,gh]} - \chi_{[gh,1]}.$$

Proof. Let $1,\ g$ and gh be vertices in the Cayley graph of $\mathcal{F}_2$. There exists a vertex m for which

$$\begin{aligned}
[1,g] &= [1,m]\,[m,g] \\
[1,gh] &= [1,m]\,[m,gh] \\
[g,gh] &= [g,m]\,[m,gh] \\
[gh,g] &= [gh,m]\,[m,g]\,.
\end{aligned}$$

We allow the possibility that $m=1,\ g$ or gh. See the diagram



Let e denote an edge. Then

$$\left(\chi_{[1,gh]}-\chi_{[gh,1]}\right)(e)=\begin{cases} 1, & \text{if } e\in[1,gh]\ \ (1) \\ -1, & \text{if } e\in[gh,1]\ \ (2) \\ 0, & \text{else}\ \ (3) \end{cases}.$$

Let $I=\chi_{[1,g]}+\chi_{[g,gh]}-\chi_{[gh,g]}-\chi_{[g,1]}$

1. Case $(1): e\in[1,gh]$. Then either $e\in[1,g]$ or $e\notin[1,g]$.

   a. If $e\in[1,g]$ then $e\in[1,m]$. Therefore,

   $$I=\chi_{[1,g]}+\chi_{[g,gh]}-\chi_{[gh,g]}-\chi_{[g,1]}=1+0+0+0=1.$$

   b. If $e\notin[1,g]$ then $e\in[m,gh]$. Therefore,

   $$I=\chi_{[1,g]}+\chi_{[g,gh]}-\chi_{[gh,g]}-\chi_{[g,1]}=0+1+0+0=1.$$

2. Case $(2): e\in[gh,1]$ is similar.

3. Case $(3):$ if $e\notin[1,gh]$ and $e\notin[gh,1]$ then $e\in[m,g]$ or $e\in[g,m]$

   (a) If $e\in[m,g]$ then $e\in[1,g]$ and $e\in[gh,g]$. In this case, $I=1-1=0$.
   (b) The case $e\in[g,m]$ is similar.

■

Remark 37 *It is interesting to note that sometimes $\xi\notin\mathcal{H}$ but $\pi_g(\xi)-\xi\in\mathcal{H}$ for every $g\in G$. In this case, $b(g)=\pi_g(\xi)-\xi$ is still a cocycle for π. Compare to example 6. We can reconstruct the cocycle given for $\mathcal{F}_2$ in this way.*

Proof. (second proof of a-T-menability of F_2).

Let $\mathcal{H}$ be the Hilbert space as above. Fix $x = 1$. Let

$$\xi(e) = \begin{cases} +\frac{1}{2}, & \text{if } e \text{ is an edge pointing to } 1 \\ -\frac{1}{2}, & \text{if } e \text{ is an edge pointing away from } 1, \end{cases}$$

so that

$$\pi_g(\xi)(e) = \begin{cases} +\frac{1}{2}, & \text{if } e \text{ is an edge pointing to } g \\ -\frac{1}{2}, & \text{if } e \text{ is an edge pointing away from } g. \end{cases}$$

Observe that $\xi \in l^\infty(E)$, but $\xi \notin l^2(E)$. Nevertheless, $b(g) = \pi_g(\xi) - \xi \in l^2(E)$ agrees with the previous cocycle since

$$\begin{aligned} b(g) &= \begin{cases} \frac{1}{2}(1-1) = 0, & \text{if } e \text{ is an edge pointing to } 1 \text{ and to } g \\ \frac{1}{2}(-1+1) = 0, & \text{if } e \text{ is an edge pointing away from } 1 \text{ and away from } g \\ \frac{1}{2}(1+1) = 1, & \text{if } e \text{ is an edge pointing away from } 1 \text{ and into } g \\ \frac{1}{2}(-1-1) = -1, & \text{if } e \text{ is an edge pointing to } 1 \text{ and away from } g \end{cases} \\ &= \begin{cases} 0, & \text{else} \\ 0, & \text{else} \\ -1, & e \in [1,g] \\ 1, & e \in [g,1]. \end{cases} \end{aligned}$$

■

3.3 Amenable groups

There are various equivalent definitions of amenability. Here we will use Folner sequence, which is the most convenient definition to work with in this context.

Definition 38 *A Folner sequence for a group G is a sequence of finite subsets F_i of G such that $\cup F_i = G$ and for $g \in G$*

$$\lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0,$$

where $|F|$ is the cardinality of F. A group is amenable if it admits a Folner sequence.

Example 39 *The infinite cyclic group $\mathbb{Z}$ is amenable. Here the sequence of intervals $F_n = \{-n, -n+1, ..., n\}$ is a Folner sequence. For $g \in \mathbb{Z}$, $gF_n = \{-n+g, ..., g, ..., n+g\}$. For $g \geq 0$ and $n > g$,*

$$\begin{aligned} |gF_n \triangle F_n| &= |(gF_n \cup F_n) - (gF_n \cap F_n)| \\ &= |\{-n, .., -n+g-1\} \cup \{n+1, .., n+g\}| \\ &= 2g \end{aligned}$$

and

$$|F_n| = 2n+1$$

so that

$$\frac{|gF_n \triangle F_n|}{|F_n|} = \frac{2g}{2n+1} \to 0 \text{ as } n \to \infty.$$

Remark 40 *An amenable group is a-T-menable. We will illustrate this by proving that the infinite cyclic group is a-T-menable. We will provide a Hilbert space and construct an explicit affine isometric action of the infinite cyclic group on the constructed Hilbert space using the Folner sequence.*

Proof. Let $\mathcal{H} = \bigoplus_{\mathbb{N}} l^2(\mathbb{Z})$. Then $\mathcal{H}$ is a Hilbert space. Let $F_i = \chi_{[-i,i]} \in l^2(\mathbb{Z})$. For $g \in \mathbb{Z}$ the left regular action of g on $\mathbb{Z}$ is

$$gF_i = \lambda_g \chi_{[-i,i]} = \chi_{[-i+g,i+g]}.$$

Define $\pi : \mathbb{Z} \to U(\mathcal{H})$ by $\pi_g = \bigoplus_{\mathbb{N}} \lambda_g$ and $b : \mathbb{Z} \to \mathcal{H}$ by

$$\begin{aligned} b(g) &= \bigoplus_{\mathbb{N}} \left(\lambda_g \frac{F_{i^2}}{|F_{i^2}|} - \frac{F_{i^2}}{|F_{i^2}|} \right) \\ &= \bigoplus_{\mathbb{N}} \left(\lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right). \end{aligned}$$

We require the following elementary estimates:

1. When $g > 2i^2 + 1$,
$$\left\| \lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right\|^2 = \frac{1}{2i^2+1} \left\| F_{[-i^2+g,i^2+g]} - F_{[-i^2,i^2]} \right\|^2 = 2.$$

2. When $g \leq 2i^2 + 1$,
$$\left\| \lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right\|^2 \leq \frac{2g}{2i^2+1}.$$

We first check that $b(g) \in \mathcal{H}$. Fix $g \in \mathbb{Z}$. Then:

$$\begin{aligned} \|b(g)\|^2 &= \left\| \bigoplus_{\mathbb{N}} \left(\lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right) \right\|^2 \\ &= \sum_{\mathbb{N}} \left\| \lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right\|^2 \\ &= \sum_{\mathbb{N}} \frac{1}{2i^2+1} \left\| \chi_{[-i^2+g,i^2+g]} - \chi_{[-i^2,i^2]} \right\|^2 \\ &\leq \sum_{1}^{g-1} \frac{1}{2i^2+1} \left\| \chi_{[-i^2+g,i^2+g]} - \chi_{[-i^2,i^2]} \right\|^2 \\ &\quad + \sum_{z}^{\infty} \frac{1}{2i^2+1} \left\| \chi_{[-i^2+g,i^2+g]} - \chi_{[-i^2,i^2]} \right\|^2 \\ &\leq C + \sum_{z}^{\infty} \frac{2g}{2i^2+1} < \infty \text{ where } C \text{ is a constant.} \end{aligned}$$

We next check properness. When $g > 2N^2 + 1$ and $i \leq N$ we have:

$$\left\| \lambda_g \frac{F_{i^2}}{\sqrt{2i^2+1}} - \frac{F_{i^2}}{\sqrt{2i^2+1}} \right\|^2 = 2,$$

for $i \leq N$, so that

$$\begin{aligned} \|b(g)\|^2 &\geq \sum_1^N \frac{1}{2i^2+1} \left\| \chi_{[-i^2+g, i^2+g]} - \chi_{[-i^2, i^2]} \right\|^2 \\ &= \sum_1^N 2 = 2N \to \infty \text{ as } N \to \infty. \end{aligned}$$

■

4 Quotients of a-T-menable groups

In this section, we give a partial answer to an interesting question: when is the quotient of an a-T-menable group by a normal supgroup a-T-menable? We shall prove that if the normal subgroup is finite, then the quotient group is a-T-menable.

Theorem 41 *Let G be an a-T-menable discrete group and N a finite normal subgroup of G. The quotient group G/N is a-T-menable.*

Proof. Since G is an a-T-menable group, there is a proper conditionally negative definite function $\psi : G \to [0, \infty)$. We define a function $\phi : G/N \to [0, \infty)$ by $\phi(x) = \frac{1}{|N|} \sum_{g \in x} \psi(g)$. We claim that this is a proper conditionally negative definite function on G/N.

To prove properness of ϕ, let $c \in [0, \infty)$. If $\phi(x) \leq c$ then $\psi(g) \leq c|N|$ for all $g \in x$. Since $|x| = |N| < \infty$, there are only finitely many such g, and hence there are only finitely many such x.

To prove that ϕ is conditionally negative definite, we need to verify that (i) $\phi(x) = \phi(x^{-1})$ for all $x \in G/N$ and (ii) if k is any natural number, $x_1, ..., x_k \in G/N$ and $a_1, ..., a_k \in \mathbb{C}$ with $\sum_i a_i = 0$, then $\sum a_i \overline{a}_j \phi(x_j^{-1} x_i) \leq 0$.

(i) Let $x \in G/N$, then

$$\begin{aligned} \phi(x^{-1}) &= \frac{1}{|N|} \sum_{g \in x^{-1}} \psi(g) \\ &= \frac{1}{|N|} \sum_{h \in x} \psi(h^{-1}) \\ &= \frac{1}{|N|} \sum_{h \in x} \psi(h) \\ &= \phi(x). \end{aligned}$$

(ii) Let $k \in \mathbb{N}$. Let $x_1, ..., x_k \in G/N$ and $a_1, ..., a_k \in \mathbb{C}$ with $\sum_i a_i = 0$. Then:

$$\sum_{i,j=1}^{k} a_i \overline{a_j} \phi(x_j^{-1} x_i) = \sum_{i,j} a_i \overline{a_j} \frac{1}{|N|} \sum_{g \in x_j^{-1} x_i} \psi(g).$$

Let $g_i N = x_i$ and $hN = x_j$; so, $h^{-1}N = x_j^{-1}$. Therefore,

$$\sum_{g \in x_j^{-1} x_i} \psi(g) = \sum_m \psi(h^{-1} g_i m),$$

and so

$$\sum_{g \in x_j^{-1} x_i} \psi(g) = \frac{1}{|N|} \sum_{n,m \in N} \psi((g_j n)^{-1} (g_i m)).$$

If follows that

$$\sum_{i,j=1}^{k} a_i \overline{a_j} \phi(x_j^{-1} x_i) = \sum_{i,j} a_i \overline{a_j} \frac{1}{|N|} \frac{1}{|N|} \sum_{n,m \in N} \psi((g_j n)^{-1} (g_i m)).$$

Let $g_i^{(n)} = g_i$ and the $a_i^{(n)} = a_i$ .We have $g_1 l, g_2 l, ..., g_k l \in G,\ l \in N$ and $a_1, ..., a_k \in \mathbb{C}$ with $\sum_i a_i = 0$. The above can be rewritten as

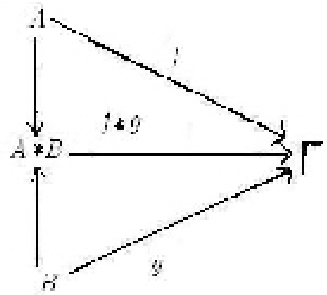
$$\begin{aligned}\sum_{i,j} a_i \overline{a_j} \phi(x_i^{-1} x_j) &= \frac{1}{|N|^2} \sum_{\substack{(g_i,n) \in \{g_1,...,g_k\} \times N \\ (g_j,m) \in \{g_1,...,g_k\} \times N}} a_i \overline{a_j} \psi((g_j n)^{-1} (g_i m)) \\ &\leq 0\end{aligned}$$

by the conditionally negative definiteness of ψ. ■

5 Free products

In this section, we will first recall some basic definitions of the free product of two groups. Then, we will prove that the free product of two groups A and B is a-T-menable whenever A and B are.

Let A and B denote groups. We form the free product G of A and B, $G = A * B$, which is characterized by the following universal property: Let Γ be a group and let f and g be homomorphisms from A and B into Γ. Then there is a unique homomorphism $f * g : A * B \to \Gamma$ making the diagram

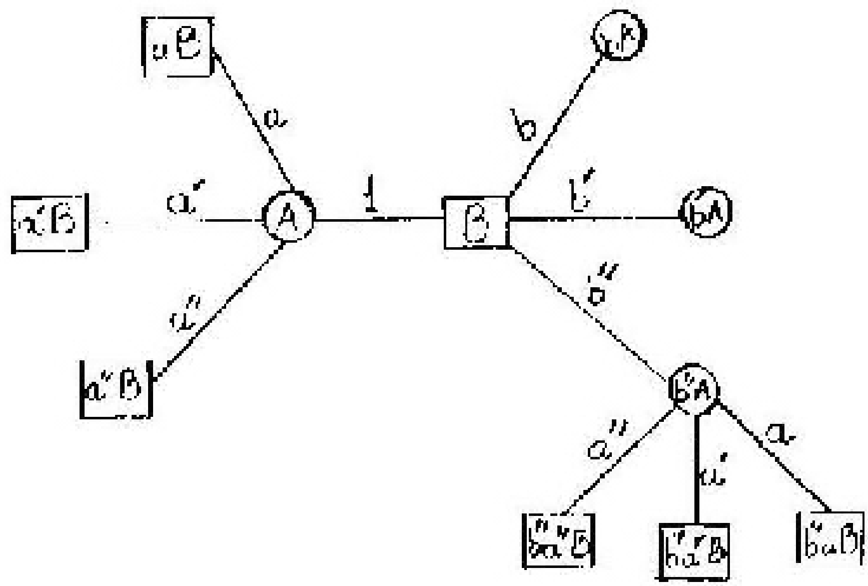


commute.

Normal form: every element of A and B is uniquely expressed as a product $a_1b_1...a_nb_n$ for some n, $a_i \in A$, $b_j \in B$ where $a_2, ..., a_n \neq e_A$; $b_1, ..., b_n \neq e_B$.

Remark 42 *Let W_A and W_B denote the set of reduced words starting with an element of A and B, respectively. Then the set of normal forms is $W_A \cup W_B$.*

Bass-Serre tree: The Bass-Serre tree is a pair (V, E) such that $V = \{gA\} \cup \{hB\}$ and edges $E = G$. Two vertices are connected by an edge when they intersect. (Observe that if $gA \cap hB \neq \emptyset$ then there is a unique $k \in gA \cap hB$). The Bass-Serre tree is represented by the following diagram



.

Notations: In what follows, we will use α, β, γ for affine actions; π, λ, ρ for unitary representations and r, s and t for cocycles.

Theorem 43 *Let A and B be groups. If A and B are a-T-menable then so is $A * B$.*

Proof. We will prove the theorem in three steps: first we will define an affine isometric action γ of G on the Hilbert space $\mathcal{H}_G$ (defined later); next we will consider the action of G on Bass-Serre tree to obtain an affine action σ; last, we will obtain the affine action as the direct sum of γ and σ.

Step 1: By hypothesis, there exist affine isometric actions α and β such that

$$\begin{aligned}\alpha &: A \to U(\mathcal{H}) \ltimes \mathcal{H};\ \alpha_a = \pi_a + r(a)\\ \beta &: B \to U(\mathcal{H}) \ltimes \mathcal{H};\ \beta_b = \lambda_b + s(b),\end{aligned}$$

where π and λ are the unitary representations of A and B respectively, and r and s are the cocycles corresponding to π and λ.

Let $\mathcal{H}_G = \bigoplus\limits_{g \in W_A \cup W_B} \mathcal{H} \otimes \delta_g$. We then obtain an affine isometric action:

$$\widetilde{\alpha} : A \to U(\mathcal{H}_G) \ltimes \mathcal{H}_G;\ \widetilde{\alpha}_a = \widetilde{\pi}_a + \widetilde{r}(a)$$

such that for $u \in \mathcal{H}$

$$\begin{aligned}\widetilde{\pi}_a(u \otimes \delta_g) &= \pi_a(u) \otimes \delta_{a \cdot g}\\ \widetilde{r}(a) &= r(a) \otimes \delta_{e_A}\end{aligned}$$

where

$$a \cdot g = \begin{cases} e_A, & \text{if } g = e_A \\ ag, & \text{if } g \neq e_A,\ g \neq a^{-1},\ g \in W_A \\ e_B, & \text{if } g = a^{-1} \\ ag, & \text{if } g \in W_B. \end{cases}$$

We verify here that $\widetilde{\pi}_a$ is a homomorphism. To do this, it is enough to verify that $a_1 \cdot (a_2 \cdot g) = (a_1 a_2) \cdot g$. Now

$$a_2 \cdot g = \begin{cases} e_A, & \text{if } g = e_A \\ a_2 g, & \text{if } g \neq e_A,\ g \neq a_2^{-1},\ g \in W_A \\ e_B, & \text{if } g = a_2^{-1} \\ a_2 g, & \text{if } g \in W_B,\ g \neq e_B \\ a_2, & \text{if } g = e_B, \end{cases}$$

and so

$$
\begin{aligned}
a_1\cdot(a_2\cdot g) &= \begin{cases} e_A, & \text{if } g=e_A\\ a_1a_2g, & \text{if } g\neq e_A,\ a_2g\neq a_1^{-1},\ g\in W_A\\ e_B, & \text{if } a_2g=a_1^{-1}\\ a_1a_2g, & \text{if } a_2g\neq a_1^{-1}, g\in W_B,\ g\neq e_B\\ \begin{cases} a_1a_2 \\ e_B\end{cases}, & \begin{matrix}\text{if } g=e_B,\ a_2\neq a_1^{-1}\\ \text{if } g=e_B,\ a_2=a_1^{-1}\end{matrix}\end{cases}\\
&= \begin{cases} e_A, & \text{if } g=e_A\\ a_1a_2g, & \text{if } g\neq e_A,\ g\neq (a_1a_2)^{-1},\ g\in W_A\\ e_B, & \text{if } g=(a_1a_2)^{-1}\\ a_1a_2g, & \text{if } g\neq (a_1a_2)^{-1}, g\in W_B,\ g\neq e_B\\ \begin{cases} a_1a_2 \\ e_B\end{cases}, & \begin{matrix}\text{if } g=e_B,\ a_2\neq a_1^{-1}\\ \text{if } g=e_B,\ a_2=a_1^{-1}\end{matrix}\end{cases}\\
&= \begin{cases} e_A & \text{if } g=e_A\\ a_1a_2g & \text{if } g\neq e_A,\ g\neq (a_1a_2)^{-1},\ g\in W_A\\ e_B & \text{if } g=(a_1a_2)^{-1}\\ a_1a_2g & \text{if } g\in W_B\end{cases}\\
&= (a_1a_2)\cdot g.
\end{aligned}
$$

We verify that $\widetilde{\pi}_a$ is unitary here. Let $u,v\in\mathcal{H}$ and $g,h\in W_A\cup W_B$ then

$$
\begin{aligned}
\langle\widetilde{\pi}_a(u\otimes\delta_g),\widetilde{\pi}_a(v\otimes\delta_h)\rangle &= \langle\pi_a(u)\otimes\delta_{a\cdot g},\pi_a(v)\otimes\delta_{a\cdot h}\rangle\\
&= \langle\pi_a(u),\pi_a(v)\rangle\langle\delta_{a\cdot g},\delta_{a\cdot h}\rangle\\
&= \langle u,v\rangle\langle\delta_g,\delta_h\rangle\\
&= \langle u\otimes\delta_g, v\otimes\delta_h\rangle,
\end{aligned}
$$

where the third equality follows from that fact that $\langle\delta_g,\delta_h\rangle=1\Leftrightarrow g=h\Leftrightarrow a\cdot g=a\cdot h\Leftrightarrow\langle\delta_{a\cdot g},\delta_{a\cdot h}\rangle=1$.

We verify the cocycle indentity holds for $\widetilde{r}$ here. Indeed, for a_1 and $a_2\in A$,

$$
\begin{aligned}
\widetilde{r}(a_1a_2) &= r(a_1a_2)\otimes\delta_{e_A}\\
&= (\pi_{a_1}(r(a_2))+r(a_1))\otimes\delta_{e_A}\\
&= \pi_{a_1}(r(a_2))\otimes\delta_{e_A}+r(a_1)\otimes\delta_{e_A}\\
&= \widetilde{\pi}_{a_1}(r(a_2)\otimes\delta_{e_A})+\widetilde{r}(a_1)\\
&= \widetilde{\pi}_{a_1}(\widetilde{r}(a_2))+\widetilde{r}(a_1).
\end{aligned}
$$

Similarly, we obtain an affine isometric action $\widetilde{\beta}$ of B on the same Hilbert space.

By the universal property of the free product of groups, we obtain a unique homomorphism

$$\gamma=\widetilde{\alpha}*\widetilde{\beta}:G=A*B\to U(\mathcal{H}_G)\ltimes\mathcal{H}_G$$

which is an affine action with the unitary part

$$\widetilde{\pi} * \widetilde{\lambda} : G = A * B \to U(\mathcal{H}_G).$$

The cocycle part $t(g) = \gamma_g(0)$ satisfies

$$\begin{aligned}
t(g) &= t(a_1 b_1 ... a_n b_n)(0) \\
&= \widetilde{\alpha}_{a_1} \widetilde{\beta}_{b_1} ... \widetilde{\alpha}_{a_n} \widetilde{\beta}_{b_n}(0) \\
&= \widetilde{\alpha}_{a_1} \widetilde{\beta}_{b_1} ... \widetilde{\alpha}_{a_n}(\widetilde{s}(b_n)) \\
&= \widetilde{\alpha}_{a_1} \widetilde{\beta}_{b_1} ... \widetilde{\beta}_{b_{n-1}}(\widetilde{\pi}_{a_n}(\widetilde{s}(b_n)) + \widetilde{r}(a_n)) \\
&= ... \\
&= \widetilde{\pi}_{a_1} \widetilde{\lambda}_{b_1} ... \widetilde{\pi}_{a_n}(\widetilde{s}(b_n)) + \widetilde{\pi}_{a_1} \widetilde{\lambda}_{b_1} ... \widetilde{\lambda}_{b_{n-1}}(\widetilde{r}(a_n)) + ... + \widetilde{\pi}_{a_1} \widetilde{s}(b_1) + \widetilde{r}(a_1),
\end{aligned}$$

where $g \in G$ of the form $g = a_1 b_1 ... a_n b_n$ with $a_i \in A$ and $b_i \in B$.

Step 2: Let $\mathcal{K} = l^2$ (the set of edges of the Bass-Serre tree). G acts on the Bass-Serre in the evident manner–by acting on cosets– and this preserves edges. An element in the basis of $\mathcal{K}$ looks like $\delta_{(gA,hB)}$ for $g, h \in G$ and $gA \cap hB \neq \emptyset$. Consider $g \in G$ of the form $g = a_1 b_1 ... a_n b_n$ ($a_1 \neq e_A$, the case when $a_1 = e_A$ is similar). Then just like the in the case of the free group on two generators $\mathcal{F}_2$, we have an affine isometric action of G on $\mathcal{K}$ with the unitary part σ and cocycle part b as follows

$$\begin{aligned}
\sigma &: G \to U(\mathcal{K}); \ \pi_g(\delta_e) = \delta_{ge} = \delta_{g(hA,kB)} = \delta_{(ghA,gkB)}; \\
b &: G \to \mathcal{K}; \ b(g) = \delta_{a_1} + \delta_{a_1 b_1} + ... + \delta_{a_1 b_1 ... a_n b_n} - \delta_{\overline{a}_1} - \delta_{\overline{a_1 b_1}} - ... - \delta_{\overline{a_1 b_1 ... a_n b_n}}.
\end{aligned}$$

We note here that all the edges without bars on top point out of the coset A and all the edges with bars on top point towards A.

Similar to the case of $\mathcal{F}_2$, σ is a unitary representation.

Step 3: Consider the Hilbert space formed by taking the direct sum of $\mathcal{H}_G$ and $\mathcal{K}$ equipped with the affine action

$$\begin{aligned}
\theta &: G \to U(\mathcal{H}_G \oplus \mathcal{K}) \ltimes (\mathcal{H}_G \oplus \mathcal{K}); \\
\theta_g &= (\gamma \oplus \sigma)_g + (t \oplus b)(g).
\end{aligned}$$

It remains to check the properness condition for $t \oplus b$. We will show that for any $g \in G$ and any number C

$$\# \left\{ g : \|t(g)\|^2 + \|b(g)\|^2 \leq C \right\} < \infty.$$

Indeed, if (‡) $g = a_1 b_1 ... a_n b_n \in G$ and for definiteness assume $a_1 \neq e$ and $b_n \neq e$ then

$$\begin{aligned}
& \|t(g)\|^2 + \|b(g)\|^2 \\
= & \sum_{i=1}^{n} \left(\|s(b_i)\|^2 + \|r(a_i)\|^2 \right) + 2(\#\text{ oriented edges}) \\
= & \sum_{i=1}^{n} \left(\|s(b_i)\|^2 + \|r(a_i)\|^2 \right) + 4n.
\end{aligned}$$

Thus,

$$\|t(g)\|^2 + \|b(g)\|^2 \le C \Rightarrow 2n \le C$$

which implies that there are at most C terms in (‡) and also for each term, $\|s(b_i)\|^2 \le C,\ \|r(a_i)\|^2 \le C$. Since there are finitely many terms and there are finitely many choices per term, we see that there are finitely many such g. ■

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